On the residual effective potential within Global 1D Quantum Gravity

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Abstract

The conjecture on Global One–Dimensionality within Quantum General Relativity leads to the model of quantum gravity possessing nontrivial field theoretic content. This is a midisuperspatial model, which quantum mechanical part can be considered independently.

The fragment, basing on the Dirac–Faddeev canonical primary quantization of Hamiltonian constraint, in fact constitutes minimal effective model within standard quantum geometrodynamics with potential different from the standard. It uses one–dimensional wave functions, where the (global) dimension is a volume form of a 3-embedding.

In this paper some elements of the global 1D quantum mechanics are presented. We consider absence of matter fields. Generalized functional expansion in the global dimension of the effective potential is discussed. Finally, its residual approximation, the Newton–Coulomb type potential, realized by all embeddings being maximally symmetric 3-dimensional Einstein manifolds is studied.

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1 Introduction

As it was shown in the last topical papers [1, 2, 3, 4, 5, 6, 7] of the author, taking into account the Global One–Dimensionality supposition within Quantum General Relativity given by the Wheeler–DeWitt quantum geometrodynamics, according to the Dirac–Faddeev Hamiltonian approach, allows to consider this model of Quantum Gravity as a bosonic classical field theory. For this field theory quantization in the Fock space of creators and annihilators with stable Bogoliubov–Heisenberg vacuum state can be done standardly, and an adequate thermodynamics of macrostates – quantum states of 3-dimensional embedded induced space – can be constructed directly. This approach results in the model of Quantum Gravity with unique quantum–statistical content.

However, the part of this full field-theoretic model of Quantum Gravity, strictly related to canonical primary quantization of the Hamiltonian constraint, can be reconsidered separately, as independent model of Quantum Gravity. In and of itself this small piece of the quantum-statistical field theory constitutes a (globally) one-dimensional quantum mechanics describing Quantum Gravity also related to any $3 + 1$ metric of General Relativity. The Quantum Mechanics looks like formally as radial-type Schrödinger wave equation, where the global dimension is generalized distance of common situation – determinant (volume form) of metric of 3-dimensional embedding.

In this paper some elements of the one-dimensional quantum-mechanical construction are discussed. Maximally symmetric 3-dimensional Einstein manifolds, those are embeddings reconstructing the Newton–Coulomb type potential within the model of Quantum Gravity, are mainly considered.

The content of this paper is as follows. First, we discuss in condensed way the standard way from the Einstein–Hilbert General Relativity with cosmological constant and the Hawking–Hartle nondynamical boundary term, by $3 + 1$ Dirac–ADM decomposition of metric, the DeWitt constraints algebra, and the Hamiltonian Dirac–Faddeev quantization of primary and secondary constraints resulting in the Wheeler–DeWitt evolution equation. For the obtained quantum geometrodynamical model of Quantum Gravity we apply the Global One–Dimensionality supposition and by global transformation of variables we reduce the Wheeler–DeWitt theory to one-dimensional quantum mechanics with an effective potential.

Received model is related to 3-dimensional embeddings. We discuss some possible physical scenarios with respect to the effective potential. The crucial subject is discussing the situation, where the generalized Newton–Coulomb potential can be obtained. In the presented model this type effective theory is obtained by any maximally symmetric Einstein 3-manifolds. We discuss generalized boundary conditions for this case.

2 Global 1D Quantum Gravity

2.1 Quantum geometrodynamics

Pseudo–Riemannian [8] manifold (M,g) given by metric $g_{\mu\nu}$, coordinates x^μ, affine connections $\Gamma^\rho_{\mu\nu}$, curvatures: Riemann $R^\lambda_{\mu\alpha\nu}$, Ricci $R_{\mu\nu}$, Ricci scalar R

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \ , \quad \Gamma_{\sigma\mu\nu} = \frac{1}{2}\left(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) \ , \quad \Gamma^\rho_{\mu\nu} = g^{\rho\sigma}\Gamma_{\sigma\mu\nu} \ , \tag{1}$$

$$R^\lambda_{\mu\alpha\nu} = \Gamma^\lambda_{\mu\nu,\alpha} - \Gamma^\lambda_{\mu\alpha,\nu} + \Gamma^\lambda_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} \ , \quad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \ , \quad R = g^{\kappa\lambda}R_{\kappa\lambda} \ , \tag{2}$$

according to Einstein [9] is a solution of General Relativity field equations[1]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 3T_{\mu\nu} \qquad , \qquad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \qquad , \tag{3}$$

where Λ is cosmological constant, and $T_{\mu\nu}$ is stress-energy tensor, arise by Palatini [10] principle used to Hilbert–Hartle–Hawking [11, 12] action

$$S[g] = \int_M d\mu_g \left\{-\frac{R}{6} + \frac{\Lambda}{3} + \mathcal{L}\right\} - \frac{1}{3}\int_{\partial M} d\mu_h K \quad , \tag{4}$$

where K is Gauss scalar curvature of spacelike boundary $(\partial M, h)$, $\mathcal{L}$ is Matter lagrangian, and $d\mu_g = d^4x\sqrt{-g}$, $d\mu_h = d^3x\sqrt{h}$ are invariant measures.

Nash embedding theorem [13, 14, 15, 16] allows using $3+1$ Dirac–ADM decomposition [17, 18, 19], by embedding metric h_{ij}, lapse N and shift N_i,

$$g_{\mu\nu} = \left[\begin{array}{cc} -N^2 + h^{ij}N_iN_j & N_j \\ N_i & h_{ij}\end{array}\right] \quad , \quad h_{ik}h^{kj} = \delta_i^j \quad , \tag{5}$$

and transforms the action (4) into the Hamiltonian form

$$S[g] = \int dt \int_{\partial M} d^3x \left\{\pi\dot{N} + \pi^i\dot{N}_i + \pi^{ij}\dot{h}_{ij} - NH - N_iH^i\right\}, \ \left(\dot{a} \equiv \frac{\partial a}{\partial t}\right). \tag{6}$$

By Gauss–Codazzi equations [20, 21, 22], nontrivial π's, and H, H^i are

$$\pi^{ij} = \sqrt{h}\left(K^{ij} - Kh^{ij}\right) \quad , \tag{7}$$

$$H = \sqrt{h}\left\{{}^{(3)}R + K^2 - K_{ij}K^{ij} - 2\Lambda - 6\varrho\right\} \quad , \quad H^i = 2\pi^{ij}_{\ \ ;j} \quad , \tag{8}$$

where ${}^{(3)}R$ is Ricci scalar of embedding, $\varrho = n^\mu n^\nu T_{\mu\nu}$ is stress-energy tensor projected onto normal vector field $n^\mu = [1/N, -N^i/N]$. Extrinsic curvature K_{ij} ($\mathrm{Tr}K_{ij} \equiv K$) is constrained with $\dot{h}_{ij}$, N, and symmetrized intrinsic covariant derivative of $N_{(i|j)}$

$$\dot{h}_{ij} = 2\left(NK_{ij} + N_{(i|j)}\right). \tag{9}$$

[1] In this article we use standardly the geometrized units system $8\pi G/3 = c = \hbar = 1$.

According to DeWitt [23] H^i are diffeomorphisms $\widetilde{x}^i = x^i + \delta x^i$ generators

$$i\left[h_{ij}, \int_{\partial M} H_a \delta x^a d^3x\right] = -h_{ij,k}\delta x^k - h_{kj}\delta x^k{}_{,i} - h_{ik}\delta x^k{}_{,j} \quad , \tag{10}$$

$$i\left[\pi_{ij}, \int_{\partial M} H_a \delta x^a d^3x\right] = -\left(\pi_{ij}\delta x^k\right)_{,k} + \pi_{kj}\delta x^i{}_{,k} + \pi_{ik}\delta x^j{}_{,k} \quad , \tag{11}$$

where $H_i = h_{ij}H^j$. Dirac [24] time-preservation of the primary constraints $\pi \approx 0$ and $\pi^i \approx 0$ leads to secondary constraints - scalar and vector

$$H \approx 0 \quad , \quad H^i \approx 0 \quad , \tag{12}$$

which create nontrivial first-class type constraints algebra [23]

$$i\left[\int_{\partial M} H\delta x_1 d^3x, \int_{\partial M} H\delta x_2 d^3x\right] = \int_{\partial M} H^a \left(\delta x_{1,a}\delta x_2 - \delta x_1 \delta x_{2,a}\right) d^3x \quad , \tag{13}$$

$$i\left[H_i(x), H_j(y)\right] = \int_{\partial M} H_a c^a_{ij} d^3z \quad , \quad i\left[H(x), H_i(y)\right] = H\delta^{(3)}_{,i}(x,y) \quad , \tag{14}$$

where $c^a_{ij} = c^a_{ij}[x,y,z] = \delta^a_i \delta^b_j \delta^{(3)}_{,b}(x,z)\delta^{(3)}(y,z) - (i \leftrightarrow j, x \leftrightarrow y)$ are structure constants of diffeomorphism group, and all Lie's brackets of π's and H's vanish. Scalar constraint determines dynamics, vector one merely reflects diffeoinvariance. By using of the conjugate momenta (7) the scalar constraint transforms into the Einstein–Hamilton–Jacobi equation widely famous in the last four decades literature [25]–[68]

$$H = G_{ijkl}\pi^{ij}\pi^{kl} + \sqrt{h}\left({}^{(3)}R - 2\Lambda - 6\varrho\right) \approx 0 \quad , \tag{15}$$

where $G_{ijkl} \equiv (2\sqrt{h})^{-1}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)$ is metric of the Wheeler–DeWitt superspace, a factor space of all C^∞ Riemannian metrics on ∂M, and a group of all C^∞ diffeomorphisms of ∂M that preserve orientation [69]. The Dirac–Faddeev primary canonical quantization method [17, 70]

$$i\left[\pi^{ij}(x), h_{kl}(y)\right] = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right)\delta^{(3)}(x,y) \quad , \tag{16}$$

$$i\left[\pi^i(x), N_j(y)\right] = \delta^i_j\delta^{(3)}(x,y) \quad , \quad i\left[\pi(x), N(y)\right] = \delta^{(3)}(x,y) \quad , \tag{17}$$

used for the Hamiltonian constraint (15) leads to the standard model of Quantum Gravity based on the Wheeler–DeWitt equation [71, 23]

$$\left\{-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - \sqrt{h}\left(-{}^{(3)}R + 2\Lambda + 6\varrho\right)\right\}\Psi[h_{ij},\phi] = 0 \quad , \tag{18}$$

called quantum geometrodynamics, where ϕ are Matter fields. Other first class constraints

$$\pi\Psi[h_{ij},\phi] = 0 \quad , \quad \pi^i\Psi[h_{ij},\phi] = 0 \quad , \quad H^i\Psi[h_{ij},\phi] = 0 \quad , \tag{19}$$

merely reflect diffeoinvariance, and are not important in this model.

2.2 The Global Dimension

The toy model of Quantum Gravity, Global One–Dimensionality supposition within the Quantum General Relativity determined by the Wheeler–DeWitt equation (18), arises from the assumption that Matter fields ϕ as well as the quantum-geometrodynamical wave function $\Psi[h_{ij},\phi]$ are functionals of only embedding 3-space volume form

$$h = \det h_{ij} = \frac{1}{3}\varepsilon^{ijk}\varepsilon^{abc}h_{ia}h_{jb}h_{kc} \quad , \tag{20}$$

where ε^{ijk} is the Levi-Civita density. So, actually the following situation

$$\phi(x) \quad \rightarrow \quad \phi[h] \quad , \tag{21}$$
$$\varrho(\phi) \quad \rightarrow \quad \varrho[h] \quad , \tag{22}$$
$$\Psi[h_{ij},\phi] \quad \rightarrow \quad \Psi[h] \quad , \tag{23}$$

lies in the fundamentals of the model. Applying the transformation of variables $h_{ij} \rightarrow h$ in the Wheeler–DeWitt equation (18), *i.e.* putting into the differential operator the relation

$$\frac{\delta}{\delta h_{ij}} = \mathcal{J}\left(h_{ij},h\right)\frac{\delta}{\delta h} \quad , \tag{24}$$

where $\mathcal{J}\left(h_{ij},h\right)$ is formally the Jacobi matrix of variables transformation

$$\mathcal{J}\left(h_{ij},h\right) = hh^{ij} \quad , \tag{25}$$

and doing elementary algebraic manipulations, the reduction of full quantum geometrodynamics – (globally) one-dimensional quantum mechanical model of Quantum Gravity is received

$$\left(\frac{\delta^2}{\delta h^2} + V_{eff}[h]\right)\Psi[h] = 0. \tag{26}$$

Here $V_{eff}[h]$ is the effective potential

$$V_{eff}[h] \equiv V_G[h] + V_C[h] + V_M[h] \quad , \tag{27}$$

that is a simple algebraic sum of the three fundamental potential constituents

$$V_G[h] = \frac{2}{3}\frac{{}^{(3)}R}{h} \quad , \quad V_C[h] = -\frac{4}{3}\frac{\Lambda}{h} \quad , \quad V_M[h] = -\frac{4}{h}\varrho[h] \quad , \tag{28}$$

related to pure geometry of 3-dimensional embedding space (G), cosmological constant (C), and Matter fields (M).

On the one side, by the simple identification of the effective potential with the square of mass of the boson $V_{eff}[h] \equiv m^2[h]$, one can state that the quantum evolution (26) describes the model of Quantum Gravity in terms of classical theory of massive bosonic field $\Psi[h]$. It leads to construction of an adequate quantum field theory in the Fock space of static Bogoliubov–Heisenberg operator basis of creators and annihilators. One can do also some statistical nature conclusions on thermodynamics of quantum states related to any 3-dimensional embedding space. The meaningful part of this field-theoretic model was discussed in the previous papers of the author [2, 3, 4, 5, 6, 7], and is not the leading theme of the present paper.

However, on the other side one can approve the nonrelativistic type interpretation of the one–dimensional quantum dynamics (26)-(28), and treat the received global model as some the effective one-dimensional Schrödinger quantum mechanics with a certain selected potential being a functional of volume form of 3-dimensional embedding space. In the spirit of this philosophy the potential $V_{eff}[h]$ has intriguing meaning – the equation (27) is the equality between any "effective physics", maybe given by other type considerations of particle physics or condensed matter physics, and three basic constituents related to an embedding 3-space – "geometric", "cosmological", and "material" ones. Let us assume that concrete form of V_{eff} can be established from any other theoretical digressions. In this case Ricci scalar curvature of a 3-dimensional embedding can be established as

$$^{(3)}R = 6\left(\varrho[h] + \frac{\Lambda}{3} + \frac{h}{4}V_{eff}[h]\right). \tag{29}$$

Immediately, however, the fundamental question suggests itself from this type construction: *How to determine the potential $V[h]$ correctly?* This theoretical problem is more sophisticated and can not be solved by direct simple way. Presently, one can list some possible physical scenarios in the one-dimensional model, with respect to the form of the potential $V[h]$.

1. The case of constant non vanishing total potential $V_{eff} = V_0 \neq 0$. For this situation the Ricci scalar curvature of an embedding and the one-dimensional wave equation are

$$^{(3)}R = 6\left(\varrho + \frac{\Lambda}{3} + \frac{V_0}{4}h\right) \quad , \quad \left(\frac{\delta^2}{\delta h^2} + V_0\right)\Psi_0[h] = 0. \tag{30}$$

   Here $\Psi_0[h]$ is the wave function related to $V_{eff} = V_0$.

2. The case of vanishing total potential $V_{eff} = 0$. For this situation the Ricci scalar curvature of an embedding and the one-dimensional wave

equation are

$$ {}^{(3)}R = 6\left(\varrho + \frac{\Lambda}{3}\right) \quad , \quad \frac{\delta^2}{\delta h^2}\Psi_F[h] = 0. \tag{31} $$

Here Ψ_F is the "free" wave function related to $V_{eff} = 0$.

3. The case, when a sum of geometric and cosmological potential contributions vanishes $V_G + V_C = 0$, but total potential is no zeroth $V_{eff} \neq 0$. For this situation the Ricci scalar curvature of an embedding and the one-dimensional wave equation are

$$ {}^{(3)}R = 2\Lambda \quad , \quad \left(\frac{\delta^2}{\delta h^2} + V_M[h]\right)\Psi_M[h] = 0. \tag{32} $$

Here Ψ_M is the "material" wave function related to $V_M \neq 0$.

4. The case, when a sum of geometric and material potential contributions vanishes $V_G + V_M = 0$, but total potential is no zeroth $V_{eff} \neq 0$. For this situation the Ricci scalar curvature of an embedding and the one-dimensional wave equation are

$$ {}^{(3)}R = 6\varrho \quad , \quad \left(\frac{\delta^2}{\delta h^2} + V_C[h]\right)\Psi_C[h] = 0. \tag{33} $$

Here Ψ_C is the "cosmological" wave function related to $V_C \neq 0$.

5. The case, when a sum of cosmological and material potential contributions vanishes $V_C + V_M = 0$, but total potential is no zeroth $V_{eff} \neq 0$. For this situation the matter fields energy density and the one-dimensional wave equation are

$$ \varrho + \frac{\Lambda}{3} = 0 \quad , \quad \left(\frac{\delta^2}{\delta h^2} + V_G[h]\right)\Psi_G[h] = 0. \tag{34} $$

Here Ψ_G is the "geometric" wave function related to $V_G \neq 0$.

6. The other, more general, proposition can be application for the effective potential $V_{eff}[h]$ the formal (functional) Laurent series expansion in volume form h in a infinitesimal neighborhood (a circle with a radius h_ϵ) of the fixed initial value of volume form h_0

$$ V_{eff}[h] = \sum_{-\infty}^{\infty} a_n (h - h_0)^n \quad \text{in} \quad C(h_\epsilon) = \{h : |h - h_0| < h_\epsilon\}, \tag{35} $$

where a_n are series coefficients given by (classical) functional integral

$$a_n = \frac{1}{2\pi i}\int_{C(h_\epsilon)} \frac{V_{eff}[h]}{(h-h_0)^{n+1}}\delta h. \tag{36}$$

In this case the Ricci scalar curvature of an embedding equals

$${}^{(3)}R = 6\left(\varrho + \frac{\Lambda}{3} + \frac{1}{4}\sum_{-\infty}^{\infty} a_n h^{n+1}\right), \tag{37}$$

and the one-dimensional wave equation (26) yields

$$\left(\frac{\delta^2}{\delta h^2} + \sum_{-\infty}^{\infty} a_n h^n\right)\Psi[h] = 0. \tag{38}$$

Naturally, there is many other opportunities for a form of the potential $V_{eff}[h]$. However, in this paper we will discuss only an especial case.

2.3 Digression on generalized dimensions

Let us note that generally the quantum mechanical equation (26) cane be reduced by more general transformation of variables

$$h \to \xi[h], \tag{39}$$

where $\xi[h]$ is any functional in the global dimension h. In this case one can rewrite the one-dimensional equation (26) as

$$\left\{\left(\frac{\delta\xi[h]}{\delta h}\right)^2 \frac{\delta^2}{\delta\xi[h]^2} + V_{eff}\left[\xi[h]\right]\right\}\Psi\left[\xi[h]\right] = 0, \tag{40}$$

and if the functional derivative in the differential operator is non zeroth then one can rewrite this equation as

$$\left\{\frac{\delta^2}{\delta\xi^2} + V[\xi]\right\}\Psi\left[\xi\right] = 0, \tag{41}$$

where the new potential $V[\xi]$ is scaled effective potential V_{eff} expressed by the generalized dimension ξ

$$V[\xi] = \left(\frac{\delta\xi[h]}{\delta h}\right)^{-2} V_{eff}\left[\xi[h]\right]. \tag{42}$$

From this consideration the following choice of the "gauge" $\xi[h]$

$$\xi[h] \equiv h, \tag{43}$$

that leads to the quantum mechanics (26) is the minimal choice. The choice of the transformation of variables in the form (43) is the simplest transformation of the kind $h_{ij} \rightarrow \xi[\det h_{ij}]$ within the Wheeler–DeWitt theory. Other, more advanced constructions, can be generated directly from this basic case, and should be justified by some physical nature's arguments. For example let us consider the following transformation of variables

$$\xi[h] = \sqrt{h}, \tag{44}$$

which can be justified by the form of the invariant measure on an 3-dimensional embedding present in the action (4) with assumption that $h > 0$. This change yields the equation (41) with the following modified effective potential

$$V[\xi] = 4\xi^2 V_{eff}[\xi]. \tag{45}$$

It cancels the singularity $\dfrac{1}{h}$, but actually causes that $V[\xi]$ must be studied with respect to the generalized dimension ξ, not the global dimension h. Further aspects of similar considerations would be studied in further papers of the author.

The very good a point of reference in searching for the generalized dimension ξ is the normalization condition of the Schrödinger quantum mechanics, which for the considered situation takes the form of a classical functional integral

$$\int_{\Omega(h_I,h)} \left|\Psi\left[\xi[h]\right]\right|^2 \delta\xi[h] = 1, \tag{46}$$

where $\Omega(h_I, h)$ is some region of integrability in a space of all 3-dimensional embeddings with metric h_{ij} and a volume form $h = \det h_{ij}$. In fact this is the main condition for possible solutions of the studied model:

Proposition. Integrability of the wave functional $\Psi[\xi[h]]$ in the sense of functional integration in the normalization condition (46) determines the generalized dimension $\xi[h]$ in the Quantum Gravity model.

The generalized dimension $\xi[h]$ can be established in the region of integrability $\Omega(h_I, h)$ as $\xi_\Omega[h_I, h]$ by using of the functional integration formula

$$\xi_\Omega[h_I, h] = \int_{\Omega(h_I,h)} \delta\xi[h]. \tag{47}$$

In this paper we will study further consequences of the simplest transformation (43). We will use standard argument which states that the normalization condition (46) establishes integrability constants of any quantum mechanical solution.

3 Maximally symmetric Einstein embeddings

Let us consider the following case in the functional expansion of the effective potential (35)

$$a_n = \begin{cases} a_{-1} = const & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}, \tag{48}$$

which follows the effective potential (27) in the Newton–Coulomb form

$$V_{eff}[h] = \frac{a_{-1}}{h}. \tag{49}$$

In this case the Ricci scalar curvature of a 3-dimensional embedding becomes

$${}^{(3)}R = 6\left(\varrho + \frac{\Lambda}{3} + \frac{1}{4}a_{-1}\right), \tag{50}$$

and the effective potential as well as the evolution (26) take the form

$$\left(\frac{\delta^2}{\delta h^2} + \frac{a_{-1}}{h}\right)\Psi[h] = 0. \tag{51}$$

In this paper we will consider the case of maximally symmetric embeddings *i.e.* the 3-dimensional manifolds with the vacuum condition

$$\varrho \equiv 0, \text{ for all values of } h. \tag{52}$$

For this situation the wave function becomes "geometric" $\Psi[h] \equiv \Psi_G[h]$, and the Ricci scalar curvature (50) simplifies to

$${}^{(3)}R = \frac{3}{2}a_{-1} + 2\Lambda \quad , \tag{53}$$

and by computation of the Ricci curvature tensor and comparison with a 3-dimensional Einstein manifold condition [72]

$$R_{ij} = \lambda h_{ij} \quad , \tag{54}$$

we obtain that the sign λ of the considered Einstein manifolds equals

$$\frac{1}{2}a_{-1} + \frac{2}{3}\Lambda = \lambda. \tag{55}$$

In general case, one can consider classification of maximally symmetric 3-dimensional Einstein manifolds (54) with respect to its sign λ (55).

Conclusion. *In Global One–Dimensional Quantum Gravity model the embeddings which are maximally symmetric 3-dimensional Einstein manifolds with the sign (55), reconstruct the Newton–Coulomb potential* $V_{eff}[h] = \dfrac{a_{-1}}{h}$.

1. *For the case of non vanishing sign* $\lambda \neq 0$ *and negative* $a_{-1} = -|\alpha|$, *the effective potential* $V_{eff}[h]$ *is Newtonian attractive potential.*

2. *For the case of non vanishing sign* $\lambda \neq 0$ *and positive* $a_{-1} = +|\alpha|$, *the effective potential* $V_{eff}[h]$ *is Coulombic repulsive potential.*

   In these cases, positiveness or negativeness of the sign λ *determines inequalities for cosmological constant* Λ *as follows*

$$\Lambda \gtrless \begin{cases} \dfrac{3}{4}|\alpha| & \text{for Newtonian case} \\ \\ -\dfrac{3}{4}|\alpha| & \text{for Coulombic case} \end{cases} \tag{56}$$

   where the inequality $\gtrless$ *is directed according to value of sign of the Einstein manifold* $\lambda \gtrless 0$.

3. *For vanishing sign* $\lambda = 0$, *one determine uniquely* $a_{-1} = \mp|\alpha| = -\dfrac{4}{3}\Lambda$.

   In this case, from the Newton law of gravitation and the Coulomb law of electrostatics we obtain the values of cosmological constant

$$\Lambda = \begin{cases} -\dfrac{9}{32\pi}m_1 m_2 & \text{for the Newton law} \\ \\ \dfrac{3}{16\pi}\dfrac{q_1 q_2}{\epsilon_0} & \text{for the Coulomb law} \end{cases} \tag{57}$$

   where geometrized units was used, $m_{1,2}$ *are masses of bodies which interact gravitationally in vacuum,* ϵ_0 *is the dielectric constant in vacuum,* $q_{1,2}$ *are values of charges interact electrically in vacuum.*

Note that in fact, by assuming the relation (36), the constant coefficient a_{-1} is equal to the Cauchy residuum of the effective potential $V_{eff}[h]$ in any fixed point h_0

$$a_{-1} = \frac{1}{2\pi i}\int_{C(h_\epsilon)} V_{eff}[h]\delta h = Res\left[\frac{2}{3h}\left({}^{(3)}R - 2\Lambda - 6\varrho\right), h = h_0\right] \quad , \tag{58}$$

and can be computed by elementary way

$$a_{-1} = \frac{2}{3}\left({}^{(3)}R - 2\Lambda - 6\varrho\right)\Big|_{h=h_0} = \frac{2}{3}{}^{(3)}R_0 - \frac{4}{3}\Lambda - 4\varrho_0 \quad , \tag{59}$$

where prefix "0" on the LHS means value in the fixed initial value of volume form h_0. Application of the relation (59) in the constraint (55) leads to the relation

$$\frac{1}{3}{}^{(3)}R_0 = \lambda \quad , \tag{60}$$

where the initial assumption of maximality $\varrho_0 \equiv 0$ was imputed. So, the studied approximation of the effective potential worthy of the title of *residual approximation*.

Let us note that, if we want to associate the residual approximation $V_{eff}[h] = \frac{a_{-1}}{h}$ with any realistic quantized Kepler problem in Newtonian or Coulombic potentials, we should put by hands the identification

$$h \equiv r \quad , \tag{61}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is a space distance in harmonic coordinates. For this case, with the formal assumption $\delta = d$, the studied evolution equation (51) becomes more familiar equation

$$\left(\frac{d^2}{dr^2} + \frac{\mp|\alpha|}{r}\right)\Psi(r) = 0 \quad , \tag{62}$$

where the number $|\alpha|$ can be taken from the Newton law of gravitation or from the Coulomb law of electricity. Of course, the obtained equation (62) looks like formally as the radial-type Schrödinger wave equation [73] with classical Newton–Coulomb potential. The assumption $\delta = d$ is well established in the context of classical mechanics [74] and continuation of this idea into quantum mechanics is a question of an analogy only.

However, there is many possible metrics h_{ij} with the same determinant r, for example we have obviously

$$h_{ij} = r^{1/3}\delta_{ij}. \tag{63}$$

However, more generally, one can parameterize the relation (61) by $SO(3)$ group rotation matrix r_{ij}: $h_{ij} = r^{1/3}r_{ij}$, which allows use the Euler angles (θ, φ, ϕ) as follows

$$r_{ij}(\theta, \varphi, \phi) \equiv r^{(3)}_{il}(\theta) r^{(2)}_{lk}(\varphi) r^{(3)}_{kj}(\phi) \quad , \tag{64}$$

where matrices $r^{(p)}_{ij}(\vartheta)$ are rotation matrices around the p-axis

$$r^{(3)}_{ij}(\vartheta) = \left[\begin{array}{ccc} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{array}\right], \qquad r^{(2)}_{ij}(\vartheta) = \left[\begin{array}{ccc} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{array}\right]. \tag{65}$$

4 Geometric wave functions

Still we will consider solutions of the one-dimensional quantum mechanics (26) for the discussed residual approximation of the effective potential $V_{eff}[h]$. For the considered case the evolution is solved by two type geometric wave functions $\Psi_G[h] \equiv \Psi_G^{\mp}[h]$

$$\left(\frac{\delta^2}{\delta h^2} \mp \frac{|\alpha|}{h}\right)\Psi_G^{\mp}[h] = 0 \quad , \tag{66}$$

where the attractive wave functions $\Psi_G^{-}[h]$ are associated with the sign "$-$" in the potential (Newtonian case), and the repulsive ones $\Psi_G^{+}[h]$ are associated with the sign "$+$" in the potential (Coulombic case). One treat the functional evolution (66) as a type of an order differential equation for wave functions $\Psi_G^{\mp}[h]$. General solution of this equation can be constructed directly in terms of the Bessel functions J_n and Y_n for the case of Newtonian attractive potential

$$\Psi_G^{-}[h] = \sqrt{|\alpha|h}\left[C_1^{-}J_1\left(2\sqrt{|\alpha|h}\right) + 2iC_2^{-}Y_1\left(2\sqrt{|\alpha|h}\right)\right] \quad , \tag{67}$$

as well as in terms of the modified Bessel functions I_n and K_n for the case of Coulombic repulsive potential

$$\Psi_G^{+}[h] = -\sqrt{|\alpha|h}\left[C_1^{+}I_1\left(2\sqrt{|\alpha|h}\right) + 2C_2^{+}K_1\left(2\sqrt{|\alpha|h}\right)\right] \quad , \tag{68}$$

where $C_1^{\pm}$ and $C_2^{\pm}$ are constants of integration, the Bessel functions of first and second kind, $J_\alpha(x)$ and $Y_\alpha(x)$, are

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi dt\cos\left(x\cos t - \alpha t\right) \quad , \tag{69}$$

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos\left(\alpha\pi\right) - J_{-\alpha}(x)}{\sin\left(\alpha\pi\right)} \quad , \tag{70}$$

and the modified Bessel functions of first and second kind, $I_\alpha(x)$ and $K_\alpha(x)$, are

$$I_\alpha(x) = \frac{1}{\pi}\int_0^\pi dt\exp\left(x\cos t\right)\cos\left(\alpha t\right) \quad , \tag{71}$$

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin\left(\alpha\pi\right)}. \tag{72}$$

Standardly, values of the second kind Bessel functions and modified ones for any integers n can be received by employing of the limiting procedure $Y_n(x) = \lim_{\alpha\to n} Y_\alpha(x)$, $K_n(x) = \lim_{\alpha\to n} K_\alpha(x)$ [75]. The main subject of this section is studying of solutions of the quantum mechanical evolution (66) with respect to boundary conditions for the general solutions (67) and (68).

4.1 Boundary conditions I

Let us consider the case of the quantum evolution (26) with the boundary values for some fixed $h = h_I$:

$$\Psi[h_I] = \Psi_I \quad , \quad \frac{\delta\Psi}{\delta h}[h_I] = \Psi'_I. \tag{73}$$

With using of the regularized hypergeometric functions ${}_p\tilde{F}_q$

$$\begin{aligned}
{}_p\tilde{F}_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right) &= \frac{{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right)}{\Gamma(b_1)\ldots\Gamma(b_q)}, &(74)\\
{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right) &= \sum_{r=0}^{\infty}\frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r}\frac{x^r}{r!}, &(75)\\
(a)_r &\equiv \frac{\Gamma(a+r)}{\Gamma(a)}, &(76)
\end{aligned}$$

one write the general solutions (67) and (68) for the considered boundary conditions (73) in the following form

$$\Psi^-_G = C^-_1\left(2\sqrt{|\alpha|h}\right)K_1\left(2\sqrt{|\alpha|h}\right) + C^-_2\left(2\sqrt{|\alpha|h}\right)^2 {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h\right), \tag{77}$$

with constans

$$\begin{aligned}
C^-_1 &= \Psi_I\, {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 1 \end{array};|\alpha|h_I\right) - \Psi'_I h_I\, {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h_I\right), &(78)\\
C^-_2 &= \frac{1}{2}\left(\Psi_I K_0\left(2\sqrt{|\alpha|h_I}\right) + \Psi'_I\sqrt{\frac{h_I}{|\alpha|}}K_1\left(2\sqrt{|\alpha|h_I}\right)\right), &(79)
\end{aligned}$$

for Newtonian case, and

$$\Psi^+_G = C^+_1\left(2\sqrt{|\alpha|h}\right)Y_1\left(2\sqrt{|\alpha|h}\right) + C^+_2\left(2\sqrt{|\alpha|h}\right)^2 {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h\right), \tag{80}$$

with constans

$$\begin{aligned}
C^+_1 &= \frac{\pi}{2}\left(\Psi'_I h_I\, {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h_I\right) - \Psi_I\, {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 1 \end{array};-|\alpha|h_I\right)\right), &(81)\\
C^+_2 &= \frac{\pi}{2}\left(\Psi_I Y_0\left(2\sqrt{|\alpha|h_I}\right) - \Psi'_I\sqrt{\frac{h_I}{|\alpha|}}Y_1\left(2\sqrt{|\alpha|h_I}\right)\right), &(82)
\end{aligned}$$

for Coulombic case.

4.2 Boundary conditions II

The second case which we want to present, are the boundary conditions for 1st and 2nd functional derivatives

$$\frac{\delta\Psi}{\delta h}[h_I]=\Psi'_I \quad , \quad \frac{\delta^2\Psi}{\delta h^2}[h_I]=\Psi''_I. \tag{83}$$

One more using hypergeometric functions, one can express the solution for attractive case as follows

$$\Psi_G^-=C_1^-\left(2\sqrt{|\alpha|h}\right)K_1\left(2\sqrt{|\alpha|h}\right)+C_2^-\left(2\sqrt{|\alpha|h}\right)^2{}_0\tilde{F}_1\left(\begin{matrix}-\\2\end{matrix};|\alpha|h\right), \tag{84}$$

where C_1^- and C_2^- are constants defined as

$$\begin{aligned} C_1^- &= -h_I\left(\Psi'_I\,{}_0\tilde{F}_1\left(\begin{matrix}-\\2\end{matrix};|\alpha|h_I\right)-\frac{\Psi''_I}{|\alpha|}\,{}_0\tilde{F}_1\left(\begin{matrix}-\\1\end{matrix};|\alpha|h_I\right)\right), && (85)\\ C_2^- &= \frac{1}{2}\sqrt{\frac{h_I}{|\alpha|}}\left(\Psi''_I\sqrt{\frac{h_I}{|\alpha|}}K_0\left(2\sqrt{|\alpha|h_I}\right)+\Psi'_I K_1\left(2\sqrt{|\alpha|h_I}\right)\right). && (86)\end{aligned}$$

Similarly for repulsive one we obtain

$$\Psi_G^+=C_1^+\left(2\sqrt{|\alpha|h}\right)Y_1\left(2\sqrt{|\alpha|h}\right)+C_2^+\left(2\sqrt{|\alpha|h}\right)^2{}_0\tilde{F}_1\left(\begin{matrix}-\\2\end{matrix};-|\alpha|h\right), \tag{87}$$

with constans

$$\begin{aligned} C_1^+ &= \frac{\pi h_I}{2}\left(\Psi'_I\,{}_0\tilde{F}_1\left(\begin{matrix}-\\2\end{matrix};-|\alpha|h_I\right)+\frac{\Psi''_I}{|\alpha|}\,{}_0\tilde{F}_1\left(\begin{matrix}-\\1\end{matrix};-|\alpha|h_I\right)\right), && (88)\\ C_2^+ &= \frac{\pi}{4}\sqrt{\frac{h_I}{|\alpha|}}\left(\Psi''_I\sqrt{\frac{h_I}{|\alpha|}}Y_0\left(2\sqrt{|\alpha|h_I}\right)+\Psi'_I Y_1\left(2\sqrt{|\alpha|h_I}\right)\right). && (89)\end{aligned}$$

4.3 Boundary conditions III

The last possible case of boundary conditions for the considered problem is

$$\Psi[h_I]=\Psi_I \quad , \quad \frac{\delta^2\Psi}{\delta h^2}[h_I]=\Psi''_I. \tag{90}$$

These boundaries are formally improper for the problem; they give singluar solutions. However, in this case one can present solutions in form with formally singular constans. For the Newtonian attractive potential the solution is

$$\Psi_G^-=C_1^-\left(2\sqrt{|\alpha|h}\right)K_1\left(2\sqrt{|\alpha|\,h}\right)+C_2^-\left(2\sqrt{|\alpha|h}\right)^2{}_0\tilde{F}_1\left(\begin{matrix}-\\2\end{matrix};|\alpha|h\right), \tag{91}$$

with constans ($\epsilon \rightarrow 0$)

$$C_1^- = \frac{2}{\epsilon}\sqrt{|\alpha|h_I}\left(\Psi_I - \frac{h_I}{|\alpha|}\Psi_I''\right){}_0\tilde{F}_1\left(\ \frac{-}{2}\ ;|\alpha|h_I\right), \tag{92}$$

$$C_2^- = \frac{1}{\epsilon}\left(\Psi_I - \frac{h_I}{|\alpha|}\Psi_I''\right)K_1\left(2\sqrt{|\alpha|h_I}\right), \tag{93}$$

and for the Coulombic repulsive potential we have

$$\Psi_G^+ = C_1^+\left(2\sqrt{|\alpha|h}\right)Y_1\left(2\sqrt{|\alpha|\,h}\right) + C_2^+\left(2\sqrt{|\alpha|h}\right)^2{}_0\tilde{F}_1\left(\ \frac{-}{2}\ ;-|\alpha|h\right), \tag{94}$$

with constants ($\epsilon \rightarrow 0$)

$$C_1^+ = \frac{2}{\epsilon}\sqrt{|\alpha|h_I}\left(\Psi_I + \frac{h_I}{|\alpha|}\Psi_I''\right){}_0\tilde{F}_1\left(\ \frac{-}{2}\ ;-|\alpha|h_I\right), \tag{95}$$

$$C_2^+ = \frac{1}{\epsilon}\left(\Psi_I + \frac{h_I}{|\alpha|}\Psi_I''\right)Y_1\left(2\sqrt{|\alpha|h_I}\right). \tag{96}$$

However, when the following condition for initial data holds

$$\pm\frac{h_I}{|\alpha|}\Psi_I^{\pm\prime\prime} + \Psi_I^{\pm} \equiv \epsilon f_{\pm}[h_I, |\alpha|], \tag{97}$$

where $f_{\pm}[h_I, |\alpha|] \neq 0$ is some (now unknown and arbitrary) nonsingular functional of h_I and $|\alpha|$, the sign $+$ is related to the Newtonian case, and the sign $-$ to the Coulombic one, then solutions (91) and (94) are nonsingular. In this case initial value of the wave function Ψ_I is for the attractive case

$$\Psi_I^- = -|\alpha|h_I\,{}_0F_1\left(\ \frac{-}{2}\ ;|\alpha|h_I\right)\left[c_1^- + 2\epsilon\sqrt{|\alpha|}\int_1^{h_I}\frac{dt}{\sqrt{t}}f_-[t,|\alpha|]K_1\left(2\sqrt{|\alpha|t}\right)\right] + \\ + 2\sqrt{|\alpha|h_I}K_1\left(2\sqrt{|\alpha|h_I}\right)\left[c_2^- + \epsilon|\alpha|\int_1^{h_I}dt f_-[t,|\alpha|]\,{}_0F_1\left(\ \frac{-}{2}\ ;|\alpha|t\right)\right], \tag{98}$$

and similarly for the repulsive one

$$\Psi_I^+ = |\alpha|h_I\,{}_0F_1\left(\ \frac{-}{2}\ ;-|\alpha|h_I\right)\left[c_1^+ - \epsilon\pi\sqrt{|\alpha|}\int_1^{h_I}\frac{dt}{\sqrt{t}}f_+[t,|\alpha|]Y_1\left(2\sqrt{|\alpha|t}\right)\right] + \\ + 2i\sqrt{|\alpha|h_I}Y_1\left(2\sqrt{|\alpha|h_I}\right)\left[c_2^+ - \epsilon\frac{i\pi}{2}|\alpha|\int_1^{h_I}dt f_+[t,|\alpha|]\,{}_0F_1\left(\ \frac{-}{2}\ ;-|\alpha|t\right)\right], \tag{99}$$

where $c_{1,2}^{\pm}$ are constants of integration. The functions $f_{\pm}[h_I, |\alpha|] \neq 0$ can be established by using of the condition (97) in general solutions (91) and (94),

it yields

$$\Psi_I^- = 8|\alpha|h_I K_1\left(2\sqrt{|\alpha|h_I}\right) {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h_I\right) f_-[h_I,|\alpha|], \quad (100)$$

$$\Psi_I^+ = 8|\alpha|h_I Y_1\left(2\sqrt{|\alpha|h_I}\right) {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h_I\right) f_+[h_I,|\alpha|]. \quad (101)$$

Now by direct application of these equations into received equalities (98) and (99) one can obtain the following integral equations for the functions $f_\pm$. For the Coulombic situation we have

$$\begin{aligned} &-\frac{c_1^-}{4}\left(2\sqrt{|\alpha|h_I}\right) {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h_I\right) + c_2^- K_1\left(2\sqrt{|\alpha|h_I}\right) + \\ &+ \epsilon|\alpha|\left\{K_1\left(2\sqrt{|\alpha|h_I}\right)\int_1^{h_I} dt\, {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|t\right) - \right. \\ &- \left.\sqrt{h_I}\, {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h_I\right)\int_1^{h_I}\frac{dt}{\sqrt{t}}K_1\left(2\sqrt{|\alpha|t}\right)\right\} f_-[t,|\alpha|] = \\ &= 2\left(2\sqrt{|\alpha|h_I}\right) K_1\left(2\sqrt{|\alpha|h_I}\right) {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};|\alpha|h_I\right) f_-[h_I,|\alpha|], \end{aligned} \quad (102)$$

and for the Newtonian one we have

$$\begin{aligned} &\frac{c_1^+}{4}\left(2\sqrt{|\alpha|h_I}\right) {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h_I\right) + ic_2^+ Y_1\left(2\sqrt{|\alpha|h_I}\right) - \\ &- \epsilon\frac{\pi}{2}|\alpha|\left\{\sqrt{h_I}\, {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h_I\right)\int_1^{h_I}\frac{dt}{\sqrt{t}}Y_1\left(2\sqrt{|\alpha|t}\right) + \right. \\ &+ \left. iY_1\left(2\sqrt{|\alpha|h_I}\right)\int_1^{h_I} dt\, {}_0F_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|t\right)\right\} f_+[t,|\alpha|] = \\ &= 2\left(2\sqrt{|\alpha|h_I}\right) Y_1\left(2\sqrt{|\alpha|h_I}\right) {}_0\tilde{F}_1\left(\begin{array}{c} - \\ 2 \end{array};-|\alpha|h_I\right) f_+[h_I,|\alpha|]. \end{aligned} \quad (103)$$

However, in both cases the integral operators acting on the functions $f_\pm$ are nonsingular. In this situation one can put the formal limit $\epsilon \to 0$ in the equations (102) and (103), and by doing some elementary algebraic manipulations one can extract the searched functions. Finally, we obtain the results

$$f_-[h_I,|\alpha|] = \frac{-c_1^-/8}{K_1\left(2\sqrt{|\alpha|h_I}\right)} + \frac{c_2^-/4}{I_1\left(2\sqrt{|\alpha|h_I}\right)}, \quad (104)$$

$$f_+[h_I,|\alpha|] = \frac{c_1^+/8}{Y_1\left(2\sqrt{|\alpha|h_I}\right)} + \frac{ic_2^+/4}{J_1\left(2\sqrt{|\alpha|h_I}\right)}. \quad (105)$$

In this manner the initial data for the studied boundary conditions (90) can not be chosen arbitrary, but according to the rules

$$\Psi_I^- = \sqrt{|\alpha|h_I}\left[-c_1^- I_1\left(2\sqrt{|\alpha|h_I}\right) + 2c_2^- K_1\left(2\sqrt{|\alpha|h_I}\right)\right], \tag{106}$$

$$\Psi_I^+ = \sqrt{|\alpha|h_I}\left[c_1^+ J_1\left(2\sqrt{|\alpha|h_I}\right) + 2ic_2^+ Y_1\left(2\sqrt{|\alpha|h_I}\right)\right]. \tag{107}$$

Of course, the supposed equation for boundary values(97) is here arbitrary, and can be replaced by other ones. However, the discussed case reflects some typical questions in the problem.

5 Vanishing sign

Finally, let us discuss briefly the case of vanishing sign (55) for studied 3-dimensional Einstein manifolds

$$\lambda = \frac{1}{2}a_{-1} + \frac{2}{3}\Lambda \equiv 0. \tag{108}$$

For this situation we have of course

$$a_{-1} = -\frac{4}{3}\Lambda \equiv \pm|\alpha|. \tag{109}$$

So, for this case in absence of Matter fields the effective potential (27) becomes purely cosmological

$$V_{eff}[h] = V_C[h] = -\frac{4}{3}\frac{\Lambda}{h}, \tag{110}$$

where the cosmological constant Λ is established according to the relation (57). From the global one-dimensional quantum mechanics point of view, the our model of Quantum Gravity defines "cosmological" wave function if and only if in the received solutions of the previous chapter we input the change

$$|\alpha| = \begin{cases} -\frac{4}{3}\Lambda & \text{for Newtonian case} \\ & \\ +\frac{4}{3}\Lambda & \text{for Coulombic case} \end{cases}, \tag{111}$$

so that the cosmological wave function is only one and can be determined by using geometric wave function with the identification (111) as follows

$$\Psi_C[h] = \Psi_G^{\pm}[h], \tag{112}$$

where the sign is chosen according to the sign of cosmological constant Λ.

6 Discussion

We have presented the quantum mechanical point of view on the Global One–Dimensional supposition within Quantum General Relativity studied previously by the author in terms of quantum field theory [1, 2, 3, 4, 5, 6, 7]. The obtained model bases on the effective potential (27) being a simple algebraic sum of three fundamental constituents - geometric, cosmological, and material, with nontrivial change in potential behavior with respect to the entrance model that was the Wheeler–DeWitt theory (18)

$$V_{eff}[h] \rightarrow h^{-3/2} V_{WDW}[h].$$

We have concentrated our attention on studying the elementary case, that we have called *residual approximation of the effective potential*, that on some conventional level $h \rightarrow r$ can be identified with the attractive Newton gravitation or the repulsive Coulomb electrostatics. Studying of this case allowed to conclude that in the case of matter fields absence, in the Global One–Dimensional model of Quantum Gravity, the maximally symmetric 3-dimensional Einstein manifolds are crucial for the residual case. Finally, we have found some solutions of the Quantum Gravity model in the residual approximation.

The digression about other possible transformations of variables $h \rightarrow \xi[h]$ (41) within the Global 1D quantum Gravity model leads to modification of the effective potential by the following way

$$V_{eff}[h] \rightarrow \left[(\xi[h])^{-3/4} \frac{\delta \xi[h]}{\delta h}\right]^2 V_{WDW}\left[\xi[h]\right],$$

and shows that the studied case of $f \equiv h$, in some conventional sense, is the simplest and can be considered as minimal quantum mechanical model within the wider field theoretic model.

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